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ORIGINAL RESEARCH

Existence results for Caputo type sequential fractional differential inclusions with nonlocal integral boundary conditions

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Abstract In this paper, we study a new class of Caputo type sequential fractional differential inclusions with nonlocal Riemann–Liouville fractional integral boundary conditions. The existence of solutions for the given problem is established for the cases of convex and non-convex multivalued maps by using standard fixed point theorems. The obtained results are well illustrated with the aid of examples.

Keywords Fractional differential inclusions · Sequential fractional derivative · Integral boundary conditions · Fixed point theorems

Mathematics Subject Classification 34A60 · 34A08 · 34B15

1 Introduction

In this paper, we investigate the existence of solutions of a sequential fractional differential inclusion:

$$({}^cD^{\alpha} + k\,{}^cD^{\alpha-1})x(t) \in F(t, x(t)), \quad t \in [0, 1], \quad 2 < \alpha \leq 3, \tag{1.1}$$

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supplemented with the boundary conditions

$$x(0) = 0, \quad x'(0) = 0, \quad x(\zeta) = a \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} x(s)ds, \tag{1.2}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α, $0 < \eta < \zeta < 1$, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and k, a, β are appropriate positive real constants.

Here, we emphasize that the integral boundary conditions (1.2) can be understood in the sense that the value of the unknown function at an arbitrary position $\zeta \in (\eta, 1)$ is proportional to the Riemann–Liouville fractional integral of the unknown function: $\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} x(s)ds$, where $\eta \in (0, \zeta)$. Further, for $\beta = 1$, the integral boundary condition reduces to the usual form of nonlocal integral condition: $x(\zeta) = a \int_0^{\eta} x(s)ds$.

Fractional differential equations have been of great interest recently. It owes to the intensive development of the theory of fractional calculus as well as its applications in various scientific fields such as physics, biomathematics, blood flow phenomena, ecology, environmental issues, viscoelasticity, aerodynamics, electro-dynamics of complex medium, electrical circuits, electron-analytical chemistry, control theory, etc. For further details, see [1–6]. Some recent results concerning fractional boundary value problems can be found in a series of papers [7–21].

There is a close connection between the sequential fractional derivatives (see page 209 in [22]) and the non sequential Riemann–Liouville derivatives [23,24]. For some recent work on sequential fractional differential equations, we refer the reader to the papers [25–27]. In [28,29], the authors studied sequential fractional differential equations with different kinds of boundary conditions. Recently, the existence of solutions for higher-order sequential fractional differential inclusions with nonlocal three-point boundary conditions has been discussed in [30]. However, to the best of our knowledge, the study of sequential fractional differential equations supplemented with nonlocal Riemann–Liouville type fractional integral boundary conditions has yet to be initiated.

The paper is organized as follows. The first result dealing with non-convex valued maps relies on a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. The second result involving convex valued maps is based on the nonlinear alternative of Leray–Schauder type while in the third result, we combine the nonlinear alternative of Leray–Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. The methods used in our analysis are well known, however their exposition in the framework of problem (1.1)–(1.2) is new.

2 Preliminaries

Let us recall some basic definitions on multi-valued maps [31,32].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0, 1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

We recall now some basic definitions of fractional calculus [1,2].

Definition 2.1 For $(n-1)-$times absolutely continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n,\ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann–Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3 Sequential fractional derivative for a sufficiently smooth function $g(t)$ due to Miller-Ross [22] is defined as

$$D^\delta g(t) = D^{\delta_1} D^{\delta_2} \ldots D^{\delta_k} g(t), \tag{2.1}$$

where $\delta = (\delta_1, \ldots, \delta_k)$ is a multi-index.

In general, the operator D^δ in (2.1) can either be Riemann–Liouville or Caputo or any other kind of integro-differential operator. For instance,

$$^cD^q g(t) = D^{-(n-q)} \Big(\frac{d}{dt}\Big)^n g(t), \quad n-1 < q < n,$$

where $D^{-(n-q)}$ is the fractional integral operator of order $n-q$. Here we emphasize that $D^{-p}f(t) = I^p f(t),\ \ p = n-q$; for more details, see page 87 [1].

To define the solution for the problem (1.1)–(1.2), we need the following lemma.

Lemma 2.4 *For $h \in C([0,1],\mathbb{R})$, the integral solution of the linear equation*

$$({}^cD^{\alpha} + k\,{}^cD^{\alpha-1})x(t) = h(t), \quad t \in [0,1],\ 2 < \alpha \le 3, \tag{2.2}$$

supplemented with the boundary conditions (1.2) is given by

$$\begin{aligned} x(t) = {} & \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)dm\right)ds \\ & -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds\Bigg] \\ & +\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds, \end{aligned} \tag{2.3}$$

where

$$\Delta = k\zeta - 1 + e^{-k\zeta} - \frac{a}{\Gamma(\beta)}\left(\frac{k\eta^{\beta+1}}{\beta(\beta+1)} - \frac{\eta^{\beta}}{\beta} + \int_0^{\eta}(\eta-s)^{\beta-1}e^{-ks}ds\right) \neq 0. \tag{2.4}$$

Proof Solving (2.2), we obtain

$$\begin{aligned} x(t) = {} & b_0e^{-kt} + \frac{b_1}{k}(1-e^{-kt}) + \frac{b_2}{k^2}(kt-1+e^{-kt}) \\ & + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds, \end{aligned} \tag{2.5}$$

where b_0, b_1, b_2 are unknown arbitrary constants. Using the boundary conditions (1.2) in (2.5), we find that $b_0 = 0$, $b_1 = 0$ and

$$\begin{aligned} b_2 = {} & \frac{k^2}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)dm\right)ds \\ & -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds\Bigg]. \end{aligned}$$

Substituting the values of b_0, b_1 and b_2 in (2.5) yields the solution (2.3). This completes the proof. □

In the sequel, we need the bounds expressed in the form of the following lemma.

Lemma 2.5 *For* $h \in C([0,1],\mathbb{R})$ *with* $\|h\| = \sup_{t\in[0,1]}|h(t)|$, *we have*

(i) $$\left|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)dm\right)ds\right| \leq \frac{\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}(\eta k + e^{-k\eta}-1)\|h\|.$$

(ii) $$\left|\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds\right| \leq \frac{\zeta^\alpha}{k\Gamma(\alpha)}(1-e^{-k\zeta})\|h\|.$$

(iii) $$\left|\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}h(\tau)d\tau\right)ds\right| \leq \frac{1}{k\Gamma(\alpha)}(1-e^{-k})\|h\|.$$

3 Existence results

Definition 3.1 A function $x \in AC^3([0,1],\mathbb{R})$ is said to be a solution of the boundary value problem (1.1)–(1.2) if $x(0)=0,\ x'(0)=0,\ x(\zeta) = a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}x(s)ds$, and there exists function $v \in S_{F,x}$ such that

$$\begin{aligned} x(t) = {}& \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)dm\right)ds \\ & -\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds\Bigg] \\ & +\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds. \end{aligned} \tag{3.1}$$

For the sake of convenience, we set

$$p = \sup_{t\in[0,1]}\left|\frac{(kt-1+e^{-kt})}{\Delta}\right| = \frac{1}{|\Delta|}(e^{-k}+k-1), \tag{3.2}$$

and

$$\Lambda = p\left\{\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta + e^{-k\eta}-1\right) + \frac{\zeta^{\alpha-1}(1-e^{-k\zeta})}{k\Gamma(\alpha)}\right\} + \frac{1-e^{-k}}{k\Gamma(\alpha)}. \tag{3.3}$$

3.1 The Lipschitz case

We prove the existence of solutions for the problem (1.1)–(1.2) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [33].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [34]).

Definition 3.2 A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called:

(**a**) γ−Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y) \text{ for each } x, y \in X;$$

(**b**) a contraction if and only if it is γ−Lipschitz with $\gamma < 1$.

Lemma 3.3 [33] *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \ne \emptyset$.*

Theorem 3.4 *Assume that:*

(A_1) *$F : [0, 1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, 1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

(A_2) *$H_d(F(t, x), F(t, \bar{x})) \le q(t)|x - \bar{x}|$ for almost all $t \in [0, 1]$ and $x, \bar{x} \in \mathbb{R}$ with $q \in C([0, 1], \mathbb{R}^+)$ and $d(0, F(t, 0)) \le q(t)$ for almost all $t \in [0, 1]$.*

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0, 1]$ if $\|q\|\Lambda < 1$, i.e.

$$\|q\|\left\{p\left[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta + e^{-k\eta} - 1\right) + \frac{\zeta^{\alpha-1}(1 - e^{-k\zeta})}{k\Gamma(\alpha)}\right] + \frac{1 - e^{-k}}{k\Gamma(\alpha)}\right\} < 1.$$

Proof Define the operator $\Omega_F : C([0, 1], \mathbb{R}) \to \mathcal{P}(C([0, 1], \mathbb{R}))$ by

$$\Omega_F(x) = \left\{\begin{array}{l} h \in C([0, 1], \mathbb{R}) : \\ h(t) = \left\{\begin{array}{l} \dfrac{(kt - 1 + e^{-kt})}{\Delta}\Bigg[a\displaystyle\int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} \\ \quad \times\left(\displaystyle\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)dm\right)ds \\ \quad - \displaystyle\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)ds\Bigg] \\ \quad + \displaystyle\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)ds, \quad \text{for } v \in S_{F,x}. \end{array}\right. \end{array}\right\}$$

Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0, 1], \mathbb{R})$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [35]). Now we show that the operator Ω_F satisfies the assumptions of Lemma 3.3. To show that $\Omega_F(x) \in$

$\mathcal{P}_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) = & \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_n(\tau)d\tau\right)dm\right)ds \\ & -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_n(\tau)d\tau\right)ds\Bigg] \\ & +\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_n(\tau)d\tau\right)ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) \to u(t) = & \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)dm\right)ds \\ & -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds\Bigg] \\ & +\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds. \end{aligned}$$

Hence, $u \in \Omega_F(x)$.

Next we show that there exists $\delta < 1$ such that

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \leq \delta\|x-\bar{x}\| \quad \text{for each } x, \bar{x} \in AC^1([0,1],\mathbb{R}).$$

Let $x, \bar{x} \in AC^1([0,1],\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = & \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(\tau)d\tau\right)dm\right)ds \\ & -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(\tau)d\tau\right)ds\Bigg] \\ & +\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(\tau)d\tau\right)ds. \end{aligned}$$

By (A_2), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq q(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq q(t)|x(t)-\bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq q(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [35]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t, \bar{x}(t))$. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq q(t)|x(t)-\bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} &\times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_2(s)d\tau\right)dm\right)ds \\ &-\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_2(\tau)d\tau\right)ds\Bigg] \\ &+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_2(\tau)d\tau\right)ds. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t)-h_2(t)| \leq \left|\frac{(kt-1+e^{-kt})}{\Delta}\right|\Bigg[|a|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} &\times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} |v_1(\tau)-v_2(\tau)|d\tau\right)dm\right)ds \\ &+\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} |v_1(\tau)-v_2(\tau)|d\tau\right)ds\Bigg] \\ &+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} |v_1(\tau)-v_2(\tau)|d\tau\right)ds \\ \leq p\Bigg[|a|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-k(s-m)}\right. \\ &\times\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} |v_1(\tau)-v_2(\tau)|d\tau\right)dm\Bigg)ds \\ &+\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} |v_1(\tau)-v_2(\tau)|d\tau\right)ds\Bigg] \end{aligned}$$

$$+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}|v_1(\tau)-v_2(\tau)|d\tau\right)ds\Bigg|$$

$$\leq \|q\|\left\{p\left[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta+e^{-k\eta}-1\right)+\frac{\zeta^{\alpha-1}(1-e^{-k\zeta})}{k\Gamma(\alpha)}\right]\right.$$
$$\left.+\frac{1-e^{-k}}{k\Gamma(\alpha)}\right\}\|x-\bar{x}\|.$$

Hence,

$$\|h_1-h_2\| \leq \|q\|\left\{p\left[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta+e^{-k\eta}-1\right)+\frac{\zeta^{\alpha-1}(1-e^{-k\zeta})}{k\Gamma(\alpha)}\right]\right.$$
$$\left.+\frac{1-e^{-k}}{k\Gamma(\alpha)}\right\}\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega_F(x),\Omega_F(\bar{x}))$$
$$\leq \|q\|\left\{p\left[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta+e^{-k\eta}-1\right)+\frac{\zeta^{\alpha-1}(1-e^{-k\zeta})}{k\Gamma(\alpha)}\right]+\frac{1-e^{-k}}{k\Gamma(\alpha)}\right\}\|x-\bar{x}\|.$$

Since Ω_F is a contraction, it follows by Lemma 3.3 that Ω_F has a fixed point x which is a solution of (1.1)–(1.2). This completes the proof. □

3.2 The upper semicontinuous case

In Theorem 3.4 the multivalued F may have convex or nonconvex values. In the case when F has convex values we can prove an existence result based on nonlinear alternative of Leray–Schauder type.

Definition 3.5 A multivalued map $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X\times Y, y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 3.6 [31, Proposition1.2] *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 3.7 [36] *Let X be a Banach space. Let $F : [0, 1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0, 1], X)$ to $C([0, 1], X)$. Then the operator*

$$\Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0, 1], X) \times C([0, 1], X)$.

For the forthcoming analysis, we need the following lemmas.

Lemma 3.8 (Nonlinear alternative for Kakutani maps)[37] *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.

Now we are in a position to prove the existence of the solutions for the boundary value problem (1.1)–(1.2) when the right-hand side is convex valued.

Theorem 3.9 *Assume that:*

(H_1) *$F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*
(H_2) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $\phi \in C([0, 1], \mathbb{R}^+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le \phi(t)\psi(\|x\|) \ \textit{for each} \ (t, x) \in [0, 1] \times \mathbb{R};$$

(H_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\|\phi\|\Lambda} > 1,$$

where Λ is defined by (3.3).

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0, 1]$.

Proof Consider the operator $\Omega_F : C([0, 1], \mathbb{R}) \to \mathcal{P}(C([0, 1], \mathbb{R}))$ defined in the beginning of the proof of Theorem 3.4. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each $x \in C([0, 1], \mathbb{R})$*. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in $C([0, 1], \mathbb{R})$*. For a positive number ρ, let $B_\rho = \{x \in C([0, 1], \mathbb{R}) : \|x\| \le \rho\}$

be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega_F(x)$, $x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$
\begin{aligned}
h(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\
&\times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)dm\right)ds \\
&-\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds\Bigg] \\
&+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds.
\end{aligned}
$$

Then for $t \in [0,1]$ we have

$$
\begin{aligned}
|h(t)| &\le \Bigg|\frac{(kt-1+e^{-kt})}{\Delta}\Bigg|\Bigg[|a|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\
&\times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}|v(\tau)|d\tau\right)dm\right)ds \\
&+\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}|v(\tau)|d\tau\right)ds\Bigg] \\
&+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}|v(\tau)|d\tau\right)ds\Bigg| \\
&\le p\Bigg[|a|\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi(\tau)\psi(\|x\|)d\tau\right)dm\right)ds \\
&+\int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi(\tau)\psi(\|x\|)d\tau\right)ds\Bigg] \\
&+\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi(\tau)\psi(\|x\|)d\tau\right)ds\Bigg| \\
&\le \psi(\|x\|)\|\phi\|\left\{p\left[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta+e^{-k\eta}-1\right)+\frac{\zeta^{\alpha-1}(1-e^{-k\zeta})}{k\Gamma(\alpha)}\right]+\frac{1-e^{-k}}{k\Gamma(\alpha)}\right\} \\
&= \psi(\|x\|)\|\phi\|\Lambda.
\end{aligned}
$$

Consequently,

$$
\|h\| \le \psi(\rho)\|\phi\|\Lambda.
$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. For each $h \in \Omega_F(x)$, we obtain

$$
\begin{aligned}
|h(t_2)-h(t_1)| \leq & \Bigg| \int_0^{t_1} \left(e^{-k(t_2-s)} - e^{-k(t_1-s)}\right)\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} v(u)du\right)ds \\
& + \int_{t_1}^{t_2} e^{-k(t_2-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} v(u)du\right)ds \Bigg| \\
& + \Bigg| \frac{k(t_2-t_1)+e^{-kt_2}-e^{-kt_1}}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\
& \times \left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)dm\right)ds \\
& - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)ds\Bigg]\Bigg| \\
\leq & \Bigg| \int_0^{t_1} \left(e^{-k(t_2-s)} - e^{-k(t_1-s)}\right)\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} \psi(\rho)\phi(u)du\right)ds \\
& + \int_{t_1}^{t_2} e^{-k(t_2-s)}\left(\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)} \psi(\rho)\phi(u)du\right)ds \Bigg| \\
& + \Bigg| \frac{k(t_2-t_1)+e^{-kt_2}-e^{-kt_1}}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\
& \times \left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \psi(\rho)\phi(\tau)d\tau\right)dm\right)ds \\
& - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \psi(\rho)\phi(\tau)d\tau\right)ds\Bigg]\Bigg|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\Omega_F : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F *is upper semicontinuous*. To this end it is sufficient to show that Ω_F *has a closed graph*, by Lemma 3.6. Let $x_n \to x_*$, $h_n \in \Omega_F(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
h_n(t) = & \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\
& \times \left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(\tau)d\tau\right)dm\right)ds \\
& - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(\tau)d\tau\right)ds\Bigg] \\
& + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(\tau)d\tau\right)ds.
\end{aligned}
$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0, 1]$,

$$h_*(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(\tau)d\tau\right)dm\right)ds - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(\tau)d\tau\right)ds\Bigg] + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(\tau)d\tau\right)ds.$$

Let us consider the linear operator $\Theta : L^1([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)dm\right)ds - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds\Bigg] + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v(\tau)d\tau\right)ds.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \Bigg\|\frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}(v_n(\tau)-v_*(\tau))d\tau\right)dm\right)ds - \int_0^\zeta e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}(v_n(\tau)-v_*(\tau))d\tau\right)ds\Bigg] + \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}(v_n(\tau)-v_*(\tau))d\tau\right)ds\Bigg\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 3.7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}v_*(\tau)d\tau\right)dm\right)ds$$

$$- \int_0^{\zeta} e^{-k(\zeta - s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(\tau)d\tau\right)ds\Bigg]$$
$$+ \int_0^{t} e^{-k(t - s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v_*(\tau)d\tau\right)ds,$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$x(t) = \lambda \frac{(kt - 1 + e^{-kt})}{\Delta}\Bigg[a \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)}$$
$$\times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)dm\right)ds$$
$$- \int_0^{\zeta} e^{-k(\zeta - s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)ds\Bigg]$$
$$+ \lambda\int_0^{t} e^{-k(t - s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} v(\tau)d\tau\right)ds.$$

Using the computations of the second step above we have

$$|x(t)| \le \psi(\|x\|)\|\phi\|\Bigg\{p\Bigg[\frac{|a|\eta^{\alpha+\beta-1}}{k^2\Gamma(\alpha)\Gamma(\beta)}\left(k\eta + e^{-k\eta} - 1\right)$$
$$+ \frac{\zeta^{\alpha-1}(1 - e^{-k\zeta})}{k\Gamma(\alpha)}\Bigg] + \frac{1 - e^{-k}}{k\Gamma(\alpha)}\Bigg\}$$
$$= \psi(\|x\|)\|\phi\|\Lambda.$$

Consequently, we have

$$\frac{\|x\|}{\psi(\|x\|)\|\phi\|\Lambda} \le 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 3.8), we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1)–(1.2). This completes the proof. □

3.3 The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [38] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0, 1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0, 1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0, 1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.10 Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0, 1] \times \mathbb{R}) \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0, 1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0, 1]\},$$

which is called the Nemytskii operator associated with F.

Definition 3.11 Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.12 [39] *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, 1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.13 *Assume that (H_2), (H_3) and the following condition holds:*

(H_4) *$F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
- *(a) $(t, x) \longmapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- *(b) $x \longmapsto F(t, x)$ is lower semicontinuous for each $t \in [0, 1]$;*

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0, 1]$.

Proof It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 3.12, there exists a continuous function $f : AC^1([0, 1], \mathbb{R}) \to L^1([0, 1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, 1], \mathbb{R})$.

Consider the problem

$$\begin{cases} ({}^cD^{\alpha} + k\,{}^cD^{\alpha-1})x(t) = f(x(t)), & t \in [0,1],\ \ 2 < \alpha \leq 3, \\ x(0) = 0,\ x'(0) = 0,\ x(\zeta) = a\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} x(s)ds,\ \beta > 0. \end{cases} \tag{3.4}$$

Observe that if $x \in AC^3([0,1],\mathbb{R})$ is a solution of (3.4), then x is a solution to the problem (1.1)–(1.2). In order to transform the problem (3.4) into a fixed point problem, we define the operator $\overline{\Omega_F}$ as

$$\begin{aligned} \overline{\Omega_F}x(t) = \frac{(kt-1+e^{-kt})}{\Delta}\Bigg[a\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \\ \times\left(\int_0^s e^{-k(s-m)}\left(\int_0^m \frac{(m-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(x(\tau))d\tau\right)dm\right)ds \\ -\int_0^{\zeta} e^{-k(\zeta-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(x(\tau))d\tau\right)ds\Bigg] \\ +\int_0^{t} e^{-k(t-s)}\left(\int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} f(x(\tau))d\tau\right)ds. \end{aligned}$$

It can easily be shown that $\overline{\Omega_F}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.9. So we omit it. This completes the proof. □

3.4 Examples

Consider the problem

$$\begin{cases} {}^cD^{3/2}(D+2)x(t) \in F(t,x(t)), & 0 \leq t \leq 1, \\ x(0) = 0, \quad x'(0) = 0,\ x(1/2) = \int_0^{1/3} x(s)ds. \end{cases} \tag{3.5}$$

Here, $\alpha = 5/2,\ k = 2,\ a = 1,\ \eta = 1/3,\ \zeta = 1/2, \beta = 1$. With the given values, we find that $\Delta \approx 0.346810,\ p \approx 3.273652,\ \Lambda \approx 0.607518$.

(i) Consider the multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$F(t,x) = \left[0, \frac{1}{3}(t+1)\sin x + \frac{2}{3}\right].$$

Then we have

$$\sup\{|v| : v \in F(t,x)\} \leq \frac{1}{3}(t+1) + \frac{2}{3},$$

and

$$H_d(F(t,x), F(t,\bar{x}) \leq \frac{1}{3}(t+1)|x-\bar{x}|.$$

Let $q(t) = \frac{1}{3}(t+1)$. Then $\|q\| = \frac{2}{3}$ and $\|q\|\Lambda \approx 0.405012 < 1$. Hence by Theorem 3.4 the problem (3.5) has a solution.

(ii) Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t,x) = \left[e^{-t}\left(\frac{|x|^3}{|x|^3+5} + 9\right), 2e^{-t}\left(\frac{|x|^3}{|x|^3+3} + 1\right) \right].$$

For $v \in F$, we have

$$|v(t)| \leq \max\left(e^{-t}\left(\frac{|x|^3}{|x|^3+5} + 9\right), 2e^{-t}\left(\frac{|x|^3}{|x|^3+3} + 1\right) \right) \leq 10e^{-t}, \quad x \in \mathbb{R}.$$

Thus

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 10e^{-t} = \phi(t)\psi(\|x\|),$$

with $\phi(t) = e^{-t}$, $\psi(\|x\|) = 10$.

By the assumption (H_3), we find that $M > 6.07518$. It follows by Theorem 3.9 that problem (3.5) has a solution.

References

1. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V, Amsterdam (2006)
3. Sabatier, J., Agrawal, O.P., Machado, J.A.T. (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
4. Tomovski, Z., Hilfer, R., Srivastava, H.M.: Fractional and operational calculus with generalized fractional derivative operators and Mittag–Leffler type functions. Integral Transf. Spec. Funct. **21**, 797–814 (2010)
5. Konjik, S., Oparnica, L., Zorica, D.: Waves in viscoelastic media described by a linear fractional model. Integral Transf. Spec. Funct. **22**, 283–291 (2011)
6. Keyantuo, V., Lizama, C.: A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications. Math. Nach. **284**, 494–506 (2011)
7. Ahmad, B., Nieto, J.J.: Riemann–Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **36**, 1–9 (2011)
8. Liang, S., Zhang, J.: Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. Math. Comput. Model. **54**, 1334–1346 (2011)
9. Su, X.: Solutions to boundary value problem of fractional order on unbounded domains in a Banach space. Nonlinear Anal. **74**, 2844–2852 (2011)
10. Bai, Z.B., Sun, W.: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369–1381 (2012)
11. Agarwal, R.P., O'Regan, D., Stanek, S.: Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. **285**, 27–41 (2012)
12. Cabada, A., Wang, G.: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**, 403–411 (2012)
13. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: A study of nonlinear fractional differential equations of arbitrary order with Riemann–Liouville type multistrip boundary conditions. Math. Probl. Eng., Art. ID 320415, 9 (2013)
14. Zhang, L., Wang, G., Ahmad, B., Agarwal, R.P.: Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. **249**, 51–56 (2013)

15. Ahmad, B., Ntouyas, S.K.: Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **20**, 1–19 (2013)
16. O'Regan, D., Stanek, S.: Fractional boundary value problems with singularities in space variables. Nonlinear Dyn. **71**, 641–652 (2013)
17. Graef, J.R., Kong, L., Wang, M.: Existence and uniqueness of solutions for a fractional boundary value problem on a graph. Fract. Calc. Appl. Anal. **17**, 499–510 (2014)
18. Wang, G., Liu, S., Zhang, L.: Eigenvalue problem for nonlinear fractional differential equations with integral boundary conditions. Abstr. Appl. Anal., Art. ID 916260, 6 (2014)
19. Ahmad, B., Agarwal, R.P.: Some new versions of fractional boundary value problems with slit-strips conditions. Bound. Value Probl. **2014**, 175 (2014)
20. Liu, X., Liu, Z.: Existence results for fractional semilinear differential inclusions in Banach spaces. J. Appl. Math. Comput. **42**, 171–182 (2013)
21. Liu, X., Liu, Z.: Existence results for fractional differential inclusions with multivalued term depending on lower-order derivative. Abstr. Appl. Anal., Art. ID 423796, 24 (2012)
22. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley and Sons, New York (1993)
23. Wei, Z., Dong, W.: Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations. Electron. J. Qual. Theory Differ. Equ. **87**, 1–13 (2011)
24. Wei, Z., Li, Q., Che, J.: Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative. J. Math. Anal. Appl. **367**, 260–272 (2010)
25. Klimek, M.: Sequential fractional differential equations with Hadamard derivative. Commun. Nonlinear Sci. Numer. Simul. **16**, 4689–4697 (2011)
26. Baleanu, D., Mustafa, O.G., Agarwal, R.P.: On L^p-solutions for a class of sequential fractional differential equations. Appl. Math. Comput. **218**, 2074–2081 (2011)
27. Bai, C.: Impulsive periodic boundary value problems for fractional differential equation involving Riemann–Liouville sequential fractional derivative. J. Math. Anal. Appl. **384**, 211–231 (2011)
28. Ahmad, B., Nieto, J.J.: Sequential fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **64**, 3046–3052 (2012)
29. Ahmad, B., Nieto, J.J.: Boundary value problems for a class of sequential integrodifferential equations of fractional order. J. Funct. Spaces Appl., Art. ID 149659, 8 (2013)
30. Ahmad, B., Ntouyas, S.K.: On higher-order sequential fractional differential inclusions with nonlocal three-point boundary conditions. Abstr. Appl. Anal., Art. ID 659405, 10 (2014)
31. Deimling, K.: Multivalued Differential Equations. Walter De Gruyter, Berlin (1992)
32. Hu, Sh, Papageorgiou, N.: Handbook of Multivalued Analysis, Theory I. Kluwer, Dordrecht (1997)
33. Covitz, H., Nadler Jr, S.B.: Multivalued contraction mappings in generalized metric spaces. Israel J. Math. **8**, 5–11 (1970)
34. Kisielewicz, M.: Differential Inclusions and Optimal Control. Kluwer, Dordrecht (1991)
35. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580. Springer, Berlin (1977)
36. Lasota, A., Opial, Z.: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13**, 781–786 (1965)
37. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2005)
38. Bressan, A., Colombo, G.: Extensions and selections of maps with decomposable values. Studia Math. **90**, 69–86 (1988)
39. Frigon, M.: Théorèmes d'existence de solutions d'inclusions différentielles, Topological Methods in Differential Equations and Inclusions, Granas A. and Frigon M. (eds) NATO ASI Series C, Vol. 472, pp. 51–87. Kluwer Acad. Publ., Dordrecht (1995)